Digital Communications

Hani Mehrpouyan¹,

Department of Electrical and Computer Engineering,
California State University, Bakersfield
Lecture 21 (Error Probability)
February 22nd, 2013

Outline

- Error Probability for QAM
- Comparison of 2-D Schemes
- Error Probability of M-ary Orthogonal Signal Sets
- Bit Error vs. Symbol Error Rate
- Gray Coding
Error Probability for QAM

- It is not easy to find exact error rate expressions for the general QAM cases.

- For instance, for the following ring QAM constellation referred to as the (4, 4, 4, 4) ring QAM, the exact error derivation is quite involved. In many cases, the error expressions found are not in closed-form.
Error Probability for QAM

For rectangular and square QAM, on the other hand, it is a trivial task to find the exact symbol error rate in AWGN.

Note that, as we discussed, rectangular QAM is really the product of two independent PAMs, e.g.,
Error Probability for QAM

- For square M-QAM, M is usually a complete square and:

\[ M\text{-QAM} = \sqrt{M}\text{-PAM} \times \sqrt{M}\text{-PAM}. \]

- We showed that the error performance of M-PAM with an average energy spending of \( E_{av} \) Joules per symbol is:

\[ P_{M\text{-PAM}}(e) = \frac{2(M - 1)}{M} Q \left( \sqrt{\frac{6E_{av}}{(M^2 - 1)N_0}} \right) \]

- For an M-QAM with an average energy spending of \( E_{av} \) Joules per symbol, we can consider two \( M^{1/2}\text{-PAMs} \) each with half that energy.

- Then, each of the above will have:

\[ P_{\sqrt{M}\text{-PAM}}(e) = \frac{2(\sqrt{M} - 1)}{\sqrt{M}} Q \left( \sqrt{\frac{3E_{av}}{(M - 1)N_0}} \right) \]
Then, the probability of correct decision for the QAM modulation will be:

\[ P_{M-QAM}(c) = P_{\sqrt{M}-PAM}(c) \times P_{\sqrt{M}-PAM}(c) = \\
[1 - P_{\sqrt{M}-PAM}(e)]^2 \]

Thus,

\[ P_{M-QAM}(e) = \\
1 - \left\{ 1 - \frac{2(\sqrt{M} - 1)}{\sqrt{M}} \right\} Q \left( \sqrt{\frac{3\mathcal{E}_{av}}{(M - 1)N_0}} \right)^2 \]

\[ = 1 - \left\{ 1 - \frac{2(\sqrt{M} - 1)}{\sqrt{M}} \right\} Q \left( \sqrt{\frac{3\log_2(M)\mathcal{E}_{av}^b}{(M - 1)N_0}} \right)^2. \]
Error Probability for QAM

- Reminder: the above only applies to the square QAM.
- To study the performance of square QAM modulation, we plot the error expressions we found above:
Comparison of 2-D Schemes

Using a very simple analysis, the following approximate symbol error rates are found:

\[ P_{M-PSK}(e) \approx 2Q \left( \frac{2E_s}{N_0} \cdot \sin \frac{\pi}{M} \right) \]

\[ P_{M-QAM}(e) \approx \frac{2(\sqrt{M} - 1)}{\sqrt{M}} Q \left( \sqrt{\frac{3}{M - 1}} \cdot \frac{E_{av}}{N_0} \right) \]

The error rates are dominated by the Q-function terms. We define:

\[ R_m \triangleq \frac{P_{M-PSK}(e)}{P_{M-QAM}(e)} \approx \frac{3/(M - 1)}{2\sin^2\left(\frac{\pi}{M}\right)} \]
Solved Problem
Comparison of 2-D Schemes

- $R_4 = 1$ as for $M = 4$, PSK and QAM are equivalent.
- For $M > 4$: $R_M > 1$, that is, QAM performs better than PSK.
- QAM uses the 2D space better as points are placed more optimally.
- Advantage of QAM over PSK for different $M$'s, i.e., different BWEs:

\[
\begin{align*}
M = 8: & \quad 10 \log_{10} R_M = 1.65 \text{ dB} \\
M = 16: & \quad 10 \log_{10} R_M = 4.20 \text{ dB} \\
M = 32: & \quad 10 \log_{10} R_M = 7.02 \text{ dB} \\
M = 64: & \quad 10 \log_{10} R_M = 9.95 \text{ dB}
\end{align*}
\]
Error Probability of M-ary Orthogonal Signal Sets

- Reminder: In orthogonal schemes the dimensionality of the signal space or constellation is equal to the number of points, i.e., \( N = M \).

- For any M-ary orthogonal scheme with equiprobable equi-energy signals (with energy \( E_s \)) the symbol error rate in AWGN with variance (power) of \( \sigma^2 \) is given by:

\[
P(e) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ 1 - [1 - Q(x)]^{M-1} \right\} \exp\left[-\frac{(x - \sqrt{E_s}/\sigma)^2}{2}\right] dx
\]
Solved Problem
Error Probability of M-ary Orthogonal Signal Sets

- The BER for different values of $M$ for orthogonal schemes are compared.
- Is this trend different from what we have seen before? Why?
- Explain! Discuss the trade-offs.
Bit Error vs. Symbol Error Rate

- Thus far, we have derived the SER for some important modulation schemes.
- In order to have a fair comparison (same footing), we would have to look at the bit SNR rather than the symbol SNR.
- Yet, we still compared the SERs rather than the BERs.
- Eventually, these are the information bits which are the object of transmission and not the signals or symbols, thus, BER must be optimized rather than the SER. The SER is intimately related to the BER.
- Let us transmit a signal $s_m(t)$ from a signal set. If the detector decides in favor of $\hat{s}_m(t) \neq s_m(t)$, then we say we have an error (one symbol or signal error, i.e., we increment the symbol error count by one).
- How many bit errors does this translate to? If we can answer this problem the problem is solved.
Bit Error vs. Symbol Error Rate

- In the above example we have: $E_s = \log_2 M \cdot E_b$ and also $T_s = \log_2 M \cdot T_b$.

- In order to find the number of erroneous bits, we need to know the binary labels for the transmitted point $s_m(t)$ and the decided signal $\hat{s}_m(t) \neq s_m(t)$. If these two binary labels differ in $d$ bit places, then we would say we have $d/\log_2 M$ of the bits at error.

- From this we conclude that the BER or bit error probability is given by the general expression:

$$P_b(e) = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} P(s_j \text{ is detected} \mid s_i \text{ is sent}) \cdot \frac{d_{ij}^H}{k}$$

- where $k = \log_2 M$ and $d_{ij}^H$ is the number of bit positions that the binary labels (addresses) for the two symbols or signals $s_i(t) \leftrightarrow s_i$ and $s_j(t) \leftrightarrow s_j$ differ.
Bit Error vs. Symbol Error Rate

Let us assume that the binary label for signal point $s_i(t)$ is a binary vector or $k$-tuple $m_i = (m_{i1}, \ldots, m_{ik})$ where $m_{ij} \in \mathbb{Z}_2 = \{0, 1\}$ (with mod 2 additions and multiplications).

**Definition:** Hamming distance $d_{ij}^H$ between two binary vectors or $k$-tuples $m_i$ and $m_j$ is defined as the number of coordinates or locations they differ at. That is:

\[
0 \leq d_{ij}^H = d^H(m_i, m_j) = \sum_{\ell=1}^{k} (m_{i\ell} \oplus m_{j\ell}) \leq k
\]

where $\oplus$ is modulo 2 (mod 2: XOR) addition operation.
Bit Error vs. Symbol Error Rate

Summary: if for \( i = 1, 2, \ldots, M \) we have the following one-to-one mappings:

\[ s_i(t) \leftrightarrow s_i \leftrightarrow m_i = (m_{i1}, \ldots, m_{ik}) \]

then:

\[
P_b(e) = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} P(s_j \mid s_i) \cdot \frac{d^H_{ij}}{k}
\]

where \( d^H_{ij} = d^H(m_i, m_j) \).

- we would like to propose mappings or labelings for signal points to minimize the bit error rate.

**Definition:** Probabilities of the form \( P(s_j \mid s_i) \) are referred to as Pairwise Error Probabilities (PEP).
Gray Coding

- In the double summation of the BER the dominant terms are naturally terms corresponding to large PEPs. Therefore, one would want to minimize the scale and effect of such terms by: making sure that points which are close in the signal space in terms of Euclidean distance are also close to each other (their labels) in terms of Hamming distance.

- This suggests the use of Gray Coding. This is merely an ordering of the 2k k-tuples such that from one sequence or label to the next one we only encounter a single bit flip.

- Gray code is actually offering more than the above. In fact, the Hamming distance between two k-tuples is proportional to their relative location or position in the ordering. It also has a wrap-around property.

- 2-bit and 3-bit Gray Coding:
### Gray Coding

<table>
<thead>
<tr>
<th>2-bit Gray coding</th>
<th>3-bit Gray coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>0 1</td>
<td>0 0 1</td>
</tr>
<tr>
<td>1 1</td>
<td>0 1 1</td>
</tr>
<tr>
<td>1 0</td>
<td>0 1 0</td>
</tr>
</tbody>
</table>

- **2-bit Gray Code**

Mirror reflection on the right:

| 1 1 0             |
| 1 1 1             |
| 1 0 1             |
| 1 0 0             |
Gray Coding

- One only needs to remember the di-bit Gray code. There is a simple recursive method to form higher length Gray codes (as shown above).

- To go from Gray code for k bits to \((k + 1)\) bits, repeat the k-bit Gray labels using a mirror reflection (see previous slide) and then add a 0 in the MSB (most significant bit) for the upper half and a 1 for the lower half.

- We will use the Gray codes to label the points of the constellation keeping in mind that we are trying to label close points using closer and more similar labels.
Gray Coding
Gray Coding

Example: the V.32 modem constellation is a square 16-QAM with the following Gray labeling.
Comparison of BER and SER

BER is given by:

\[ P_b(e) = \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} P(s_j \text{ is detected} \mid s_i \text{ is sent}) \cdot \frac{d_{ij}^H}{k} \]

where we have

\[ P_b(e) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} P(s_j \text{ is detected} \mid s_i \text{ is sent}) = P_s(e) \]

That is BER is capped with the SER. In binary cases they are obviously equal.
Gray Coding

- The exact calculation of the BER is usually very involved.

- If one does not want to deal with the cumbersome calculations required in the above expression, the following approximation can be used ($k > 1$):

\[ P_b(e) \approx \frac{1}{2} P_s(e) \]

- The above approximation is almost exact for orthogonal signal sets.
Solved Problem