Functional Transformation of the Modified CRLB

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Abstract

In this letter it is shown that similar to the Cramér-Rao lower bound (CRLB), the functional transformation property also holds for the *modified CRLB (MCRLB)*.

Index Terms

Modified Cramér-Rao Lower Bound (MCRLB) and functional transformation of MCRLB

I. Introduction

As the lower bound on the variance of unbiased estimators, the *Cramér-Rao lower bound (CRLB)* has been extensively applied in the literature for assessing the performance of estimators [1], [2]. In addition, through the functional transformation of the CRLB [2], the CRLB for the estimation of a parameter, e.g., α, can be easily extended to any function of that parameter, e.g., $g(\alpha)$. Such a useful relationship allows for the CRLB to be easily extended to any function of a fundamental parameter, e.g., lower bounding the signal power estimation when the CRLB for the estimation of the signal amplitude is already derived [2].

Even though, the CRLB has been an effective tool, it is difficult to derive the exact CRLB in scenarios when the observation is affected by other unwanted parameters in addition to the parameter of interest [3], [4]. More specifically, for applications to communication signals, evaluation of the CRLB is complicated by unknown nuisance parameters, e.g., the problem of determining the CRLB for frequency offset estimation for a signal received in white Gaussian noise in the presence of unknown phase offset. In order to overcome this, a more easily evaluated *modified CRLB (MCRLB)* is proposed in [3], though it is a looser bound [4]. As a matter of fact, the MCRLB has been effectively applied as a lower bound in many different application such as as communication, signal processing, aerospace, and more [4]–[8].

In this letter, we demonstrate that MCRLB of a parameter α can be straightforwardly extended to any function of that parameter $f(\alpha)$.

Notation: Bold face small letters, e.g., $\mathbf{x}$, are used for vectors, $|\cdot|$ is the absolute value operator, and $\ln(\cdot)$ represents the natural logarithm function.

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II. MCRLB Under Functional Transformation

In this section we derive the MCRLB for a parameter $\beta = f(\lambda)$ whose probability distribution function (pdf) is parameterized by λ.

Recall that CRLB for estimation of a parameter, λ, can be expressed as [2]

$$\text{CRLB}(\lambda) \triangleq -\frac{1}{\mathbb{E}_{\mathbf{r}}\left[\frac{\partial^2 \ln p(\mathbf{r}|\lambda)}{\partial \lambda^2}\right]} = \frac{1}{\mathbb{E}_{\mathbf{r}}\left[\left(\frac{\partial \ln p(\mathbf{r}|\lambda)}{\partial \lambda}\right)^2\right]}, \tag{1}$$

where λ is the estimated parameter, $\mathbf{r} \triangleq [r(0), \cdots, r(N-1)]^T$ is the observation sequence, and $\mathbb{E}\{\cdot\}$ denotes the expectation with respect to probability distribution function (pdf), $p(\mathbf{r}, \lambda)$. For (1) to hold, the pdf, $p(\mathbf{r}, \lambda)$, must satisfy the regularity condition [2], [9][1]

$$\mathbb{E}_{\mathbf{r}}\left[\frac{\partial \ln p(\mathbf{r}|\lambda)}{\partial \lambda}\right] = 0. \tag{2}$$

Now suppose instead that the PDF $p(\mathbf{r}|\lambda)$ is also dependent on a set of extraneous parameters $\mathbf{u}$, expressed as $p(\mathbf{r}|\lambda, \mathbf{u})$, then to determine the CRLB(λ), $p(\mathbf{r}|\lambda, \mathbf{u})$ needs to be first averaged over the unwanted parameters via

$$p(\mathbf{r}|\lambda) = \int_{\infty}^{-\infty} p(\mathbf{r}|\lambda, \mathbf{u}) p(\mathbf{u}) d\mathbf{u}. \tag{3}$$

Determining the CRLB is further complicated in this case, since to evaluate (3), assumptions regarding the distributions of the nuisances parameters, $\mathbf{u}$, need to be made. In addition, due to the fact that (3) is a multi-dimensional integration, evaluating it, unfortunately is not easy for most distributions. Thus, to avoid carrying out (3), the following modified CRLB, where from here on is denoted as the MCRLB, is given by

$$\text{MCRLB}(\lambda) = \frac{1}{\mathbb{E}_{\mathbf{r},\mathbf{u}}\left[\left(\frac{\partial \ln p(\mathbf{r}|\lambda,\mathbf{u})}{\partial \lambda}\right)^2\right]}\ , \tag{4}$$

is proposed in [3].

Let us consider all unbiased estimators, i.e., those for which

$$f(\lambda) = \mathbb{E}_r[\hat{\beta}] = \beta. \tag{5}$$

[1] note that strictly speaking there are four regularity conditions that (2) arises from [9]

After differentiating both sides

$$\begin{aligned}\frac{\partial f(\lambda)}{\partial \lambda} &= \int \hat{\beta} \frac{\partial p(\mathbf{r}|\lambda)}{\partial \lambda} \partial \mathbf{r} \\ &= \int \int \hat{\beta} \frac{\partial p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{u}) \partial \mathbf{r} \partial \mathbf{u} \\ &= \mathbb{E}_{\mathbf{u}} \left[\int \hat{\beta} \frac{\partial p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} \partial \mathbf{r} \right] \\ &= \mathbb{E}_{\mathbf{u}} \left[\int \hat{\beta} \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{r}|\lambda, \mathbf{u}) \partial \mathbf{r} \right]. \end{aligned} \tag{6}$$

In addition, due to the regularity condition (2), we have

$$\begin{aligned}\mathbb{E}_{\mathbf{u}} \left[\int \beta \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{r}|\lambda, \mathbf{u}) \partial \mathbf{r} \right] &= \int \beta \frac{\partial \ln p(\mathbf{r}|\lambda)}{\partial \lambda} p(\mathbf{r}|\lambda) \partial \mathbf{r} \\ &= \beta \mathbb{E}_{\mathbf{r}} \left[\frac{\partial \ln p(\mathbf{r}|\lambda)}{\partial \lambda} \right] = 0. \end{aligned} \tag{7}$$

Using (7), (6) can be rewritten as

$$\frac{\partial f(\lambda)}{\partial \lambda} = \mathbb{E}_{\mathbf{u}} \left[\int (\hat{\beta} - \beta) \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{r}|\lambda, \mathbf{u}) \partial \mathbf{r} \right]. \tag{8}$$

Squaring both sides

$$\left(\frac{\partial f(\lambda)}{\partial \lambda} \right)^2 \leq \mathbb{E}_{\mathbf{u}} \left[\int (\hat{\beta} - \beta) \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{r}|\lambda, \mathbf{u}) \partial \mathbf{r} \right]^2, \tag{9}$$

where the inequality in (9) follows from Jensen's inequality and the convexity of $\varphi(x) = x^2$. The Cauchy-Schwarz inequality can be applied to (9)

$$\begin{aligned}\left(\frac{\partial f(\lambda)}{\partial \lambda} \right)^2 &\leq \mathbb{E}_{\mathbf{u}} \left[\int (\hat{\beta} - \beta) \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} p(\mathbf{r}|\lambda, \mathbf{u}) \partial \mathbf{r} \right]^2 \\ &= \mathbb{E}_{\mathbf{u}} \left[\mathbb{E}_{\mathbf{r}|\mathbf{u}} \left[(\hat{\beta} - \beta) \frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} \right] \right]^2 \\ &\leq \mathbb{E}_{\mathbf{u}} \left[\mathbb{E}_{\mathbf{r}|\mathbf{u}} \left[(\hat{\beta} - \beta)^2 \right] \mathbb{E}_{\mathbf{r}|\mathbf{u}} \left[\left(\frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} \right)^2 \right] \right] \\ &= \mathrm{var}(\hat{\beta}) \mathbb{E}_{\mathbf{u},\mathbf{r}} \left[\left(\frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} \right)^2 \right]. \end{aligned} \tag{10}$$

From (10), the variance of $\hat{\beta}$ can be lower bounded as

$$\mathrm{MCRLB}(\hat{\beta}) = \frac{\left(\frac{\partial f(\lambda)}{\partial \lambda} \right)^2}{\mathbb{E}_{\mathbf{r},\mathbf{u}} \left[\left(\frac{\partial \ln p(\mathbf{r}|\lambda, \mathbf{u})}{\partial \lambda} \right)^2 \right]}, \tag{11}$$

which is the required generalization of the MCRLB of λ given in (4).

III. Conclusions

In this latter it is shown that similar to the CRLB, the MCRLB for the estimation of parameter α can be extended to any function of it, $f(\alpha)$.

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